**Topic:** 19th Century Mathematician

**Notes on Topic:** As you may have noticed, each century has had a different emphases, “their own different directions to the flow of mathematical thought”

Eighteenth century was the “century of Euler”

The nineteenth century, by contrast, had no one mathematician, but a plethora of mathematicians pushing frontiers

It was a century of abstraction, generalization, deep analysis of logical foundation of mathematics

19th century mathematics seemed to drift away from the constraints of the real world, bringing us things such as non-Euclidean geometry

When Eugenio Beltrami discovered that non-Euclidean geometry was as logically consistent as Euclid’s own, a bridge was crossed. Which geometry was used to describe the natural world was not as relevant as the logic that defined such geometry

“To a mathematician engrossed in the strange and beautiful theorems of non-Euclidean geometry, the beauty was enough.”

Simultaneously, painting was declaring its independence from visual reality, all while mathematics was being liberated from the constraints of the physical world

Starting with the development of non-Euclidean geometry, mathematicians starting carrying their studies further and further from a contact with the real world

There has been much criticism on the new mathematics that has peeled away from natural science

Mathematics historian Morris Kline wrote,

Having formulated the abstract theories, mathematicians turned away from the original concrete fields and concentrated on the abstract structures. Through the introduction of hundreds of subordinate concepts, the subject has mushroomed into a welter of smaller developments that have little relation to each other or to the original concrete fields.

Suggesting that mathematics liberation from physics has ventured too far from its roots and thus become self-indulgent and self-serving.

To contrast, even the most abstract and physically bending non-Euclidean geometry has found its way into the theories of relativity of today’s cosmologies rely heavily on a non-Euclidean model of the universe. This was not discovered until later, and certainly was an unexpected result, certainly not expected by the mathematicians who investigated the subject for its own sake.

Now non-Euclidean geometry forms part of applied mathematics necessary for inclusion in the physicist’s tool kit.

“...historians may look at today’s mathematics as having ventured too far from its ties to the real world. But it is inconceivable that mathematics will ever assume a role entirely subservient to the needs of the other sciences. Mathematical freedom will forever be the legacy of the nineteenth century.” JTG 248

While these issues were sparked due to non-Euclidean geometry, there were also issues arising over the logical foundations of calculus

Calculus foundations were pinned down by Newton and Leibniz in the seventeenth century and exploited heavily in the eighteenth century by Euler, yet these greats had not paid adequate attention to the underpinnings of calculus.

The problem lied in the use of “infinitely large” and “infinitely small”

One of the key ideas of calculus lies within the “limit” , differential and integral calculus, series convergence and continuity of functions rest upon this notion

To talk about this term “limit” and try and make this idea logically precise, difficulties instantly arise

Newton tried to examine what happens when a ratio of quantities both emerge zero simultaneously, he used the term “ultimate ratio” to describe this phenomenon, although he was referring to the limit

Leibniz also discussed the limit as a topic, but he looked at quantities of being “infinitely small” he meant quantities that are not zero, but cannot be made any smaller

Gradually the mathematics community had to address this problem that the calculus depends so heavily

Thus the nineteenth century found itself cleaning the foundational mess the eighteenth century had left behind

The process of refining the idea of a “limit” was an excruciating one, requiring precision and an appreciation of the nature of the real numbers

In 1821, French mathematician Augustin-Louis Cauchy had proposed the definition,

When the values successively attributed to a particular variable approach indefinitely a fixed value, so as to end by differing from it by as little as one wishes, this latter is called the limit of all the others.

Cauchy’s definition avoided terms such as “infinitely small” , nor did it deal with what happens at the precise moment that the limit is reached, but it dealt with a fixed value representing the limit if the variable in question differs from said value by an amount as small as we would like.

This definition removed philosophical barriers as to what happened at the moment of reaching the limit, to Cauchy this issue was irrelevant if we could get as close as we wanted to this limit value.

Cauchy’s definition was so influential that it was used in proving the major theorems of calculus.

Yet even this statement needed some fine tuning.

This definition deals with vague ideas such as “approach” and “move towards”, if we are going to rely on such vague terms, why not rely on the vague term of the “limit” itself?

Cauchy’s use of the term “indefinitely” proves to be quite vague as well.

His definition, as well, was too “wordy”, this needed to be refined to clear, concise, unambiguous symbols.

The final “arithmetization of calculus” was given by German, Karl Weierstrass and his disciples.

For the school of Weierstrass, to say that “L is the limit of the function f(x) as x approaches a” meant precisely,

Weierstrass’ definition lacked some of the charm of his predecessors, but it was mathematically [and logically] sound

This definition, as I am sure you are familiar, is still used today.

As we are familiar, there are infinitely many rational and irrational number.

And, if you take any two rational numbers, there are infinitely many irrational numbers between the two. If you take any two irrational numbers, there are infinitely many rational numbers between the two.

For a long time mathematicians felt these two groups of numbers carried the same weight composing the real numbers.

As the nineteenth century progressed, all evidence to the contrary arised

For example: a function exists such that it is continuous at each irrational number, and discontinuous at each rational number. But there does not exist a function that is continuous at each rational number, and discontinuous at each irrational number.

It is a clear indicator that there are not symmetries between the two groups of numbers, and they are certainly not interchangeable

Cauchy and Weierstrass were successful in building the calculus foundations using the “limit” but mathematicians were realizing that some of the most important building blocks of calculus were based on the idea of the *set*

The man who would take this problem and in turn, develop [single-handedly] set theory was the genius Georg Ferdinand Philip Cantor

Cantor, born in Russia in 1845, moved to Germany when he was 12

Due to religious family background, Cantor developed an interest in theology and particularly the nature of the infinite

In 1867 he completed his doctorate at the University of Berlin, he had studied with Weierstrass and developed rigor in the subject of calculus

Cantor’s research into the finer points of mathematical analysis led him to consider the differences among various sets of numbers

He set out to find the size of different sets, not through the means of counting, but through the means of finding a one-to-one correspondence between the two sets

Cantor defined this as such,

Two sets M and N are equivalent … if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.

In modern speak, the two sets have the same cardinality if they meet Cantor’s definition of equivalence above.

This definition is critical because it does not limit the sets to finite ones

This was uncharted territory, since dealing with infinite was always done so under a hostile eye

Before Cantor, mathematicians dealt with the “potential infinite”

There was refusal to look at infinite sets as ever being “completed”

But not for Cantor, he was willing to view infinite sets as being a completed entity and self-contained, to be compared with other infinite sets of objects

To Cantor, it was a solid, mathematical concept worthy of examination

**For Example:** If we look at and the set of natural numbers and the set of all even integers. By Cantor’s definition it is easy to see that N and E are equivalent because we can establish a one to one correspondence between the two. We can represent each element of N by *n* then every element of E can be written *2n* . Thus setting up a one to one correspondence.

**For Example:** If we use N and representing all integers. Can we manufacture a formula to create a one-to-one correspondence?

**Answer:**

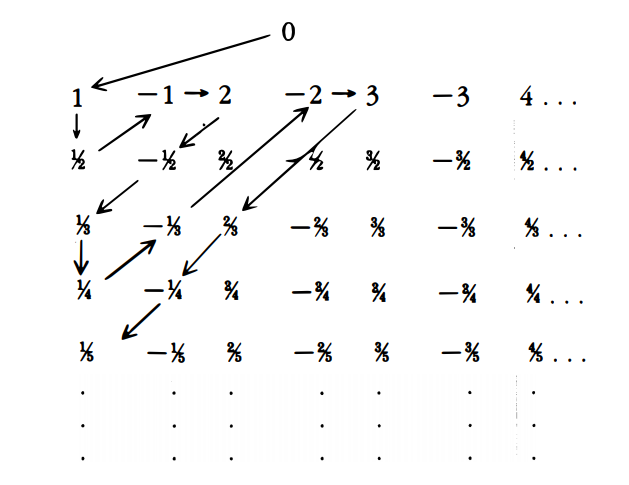
At this point, Cantor was ready to make a bold move. He said that any set that had a one-to-one correspondence with N was said to be *denumerable* or *countably infinite*

He established a transfinite cardinal number to represent the number of items in a denumerable set, (read aleph-naught)

Next let’s examine the relationship between N and Q (the set of rational numbers)

Since between any two integers there are infinitely many rational numbers, then it seems that the rationals are more plentiful than the natural numbers.

But Cantor showed the rational numbers are denumerable by conjuring a one-to-one correspondence between the two sets



Note: all numbers in the first column have numerator of 1, and all in the second column have numerator -1. All the numbers in the first row have denominator of 1, all in the second row have the denominator of 2. So to find the placement of a fraction such as 133/191, one must go down to the 191st row, and over to the 265th column (counting positive and negative numbers).

Note: we skip over any fraction that already appeared

At this point, it is clear to see that the set of rationals is denumerable

Which brings up the question, are all infinite sets denumerable, if we are able to find a one-to-one correspondence from the rationals to the natural numbers it seems so.

There was one infinite set of numbers that Cantor was able to show is non-denumerable.

In his scholarly article, *Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*, (translated “On the Property of the Collection of all Algebraic Numbers”) Cantor’s set he found that was non-denumerable was the collection of all real numbers.

Cantor showed that no matter how small the interval of real numbers, there is no one-to-one correspondence that can be established from the natural numbers to the real numbers.

**The Great Theorem:** The non-denumerability of the continuum.

Note: “continuum” means an interval of real numbers

\*\*suggested reading: The proof of this\*\*

End result: Cantor shows that the interval (0,1) is non-denumerable, therefore making the continuum non-denumerable.

Cantor’s paper also showed that the unity of two denumerable sets is also denumerable.

Cantor’s paper also tackled transcendental numbers and algebraic numbers.

Euler was the first to propose that not all numbers are well mannered and algebraic

Lindemann proved that is a transcendental number

Cantor proved the set of algebraic numbers are denumerable

Cantor wanted to examine the transcendentals, he looked at an interval (a,b) , he had already proven that the algebraic numbers within this set are denumerable, and so if the transcendentals within this interval are denumerable, then the interval (being the unity of the algebraic and transcendentals) would have to be denumerable. But Cantor proved that this such interval is non-denumerable, thus forcing the transcendentals to be non-denumerable.

Cantor proved that the transcendentals are non-denumerable, without producing even one example of a transcendental number!

In another great theorem from Chapter 12, Cantor also dealt with power sets

Given the power set of a set **A**, then P[A] is the set of all subsets of **A**.

Theorem: If **A** is any set, then

Thus concludes our study of Cantor, all his predecessors, and the history of mathematics.

**Additional Suggested Reading**: Cantor’s proof of the great theorem

**Assignment:** None